Solution of Stochastic Optimal Control Problem and its Financial Applications

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Abstract

Stochastic optimal control problems frequently occur in Economics and Finance. In this paper, we investigate stochastic optimal control problems and Bellman's dynamic programming method or Hamilton-Jacobi-Bellman (HJB) equation. Dynamic programming method represents the most known method for solving optimal control problems. Also, optimal control problems with linear control are study in this work. In this case, the HJB equation effectively breaks down into two differential equations. In fact, we propose a method to solve such problems. Finally, we simulate a financial example to highlight the applications of stochastic optimal control problems.

Keywords: Stochastic Optimal Control Problems; Hamilton-Jacobi-Bellman Equation; Financial Applications.

1 Introduction

Optimal control's models play a prominent role in a range of application areas, including aerospace, chemical engineering, robotic, economics and finance. It deals with the problem of finding a control law for a given system such that a certain optimality criterion is achieved. A controlled process is the solution to an ordinary differential equation which some parameters of the ordinary differential equation can be chosen. Hence, the trajectory of the solution is obtained. Each trajectory has an associated cost, and the optimal control problem is to minimize this cost over all choices of the control parameter. Stochastic optimal control is the stochastic extension of this; In fact, a stochastic differential equation with a control parameter is given. Each choice of the control parameter yields a different stochastic process as a solution to the stochastic differential equation. Each path wise trajectory of this stochastic process has an associated cost, and we seek to minimize the expected cost over all choices of the control parameter. In describing a stochastic control model, the kind of information available to the controller at each instant of time, plays an important role. Several situations are possible [1]:

(1) The controller has no information during the system operation. Such controls are often called open loop.

(2) The controller knows the state of the system at each instant of time t. We call this, the case of complete observations.

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Pontryagin's maximum principle method and Bellman's dynamic programming method (Hamilton-Jacobi Bellman equation) represent the most known methods for solving optimal control problems [1–7]. Pontryagin's Maximum Principle is used to find the necessary conditions for the existence of an optimum. This converts the original optimal control problem into a boundary value problem. Another very efficient approach for solving optimal control problems is dynamic programming method. It is a robust approach for solving optimal control problems. The method was originated by R. Bellman in early 1950s. Its basic idea is to consider a family of optimal control problems with different initial times and states, to establish relationships among these problems via the so-called Hamilton-Jacobi-Bellman (HJB) equation. If the HJB equation is solvable, then one can obtain an optimal feedback control by taking the maximize or minimize involved in the HJB equation [6, 7]. Unfortunately, the HJB equation is difficult to solve analytically. Thus, finding a numerical solution is at least the most logical way to treat it (for example see [8]). Two classical approaches in solving HJB equation numerically are finite differences and finite state Markov chain approximation, to reduce the problem to a finite dimensional problem, which can be solved by using vector matrix operations [9]. Both approaches however have their limitations and do not work well in the case where the problem is linear in the control [10]. In this cases optimal strategy switches between two modes, a maximum and a minimum control mode. The Hamilton-Jacobi-Bellman equation effectively breaks down into two differential equations. These equations are linked at the threshold, where it is optimal to switch [10].
This paper is organized into following sections of which this introduction is the first. In Section 2, we introduce deterministic control theory. Section 3 is about stochastic differential equations. Stochastic optimal control is presented in section 4. Section 5 is about dynamic programming; Hamilton-Jacobi-Bellman equation. In section 6, we present main results with a finantial example, This example illustrate applications of stochastic optimal controls. Finally, the paper is concluded with some keypoints of this method.

2 Deterministic Control Theory

Optimal control deals with the problem of finding a control function u. It is assumed to be piecewise from class of admissible controls, U. Each choice of control $u(.) \in U \subset \mathbb{R}^m$ yields a process $x(t) \in \mathbb{R}^n$ which is the unique solution of

$$\dot{x}(t) = f(t, x(t), u(t)), \tag{2.1}$$

which is called the equation of motion, on a fixed interval $[s, T]$ with initial condition

$$x(s) = y. \tag{2.2}$$

Along with this controlled process, there is a cost functional of the form:

$$J(s, y; u) = \psi(T, x(T)) + \int_s^T L(t, x(t), u(t))dt, \tag{2.3}$$

Here, $L(t, x, u)$ is the running cost, and $\psi(t, x)$ is the terminal cost. This cost functional depends on the initial position (s, y) and the choice of control $u(.)$. The optimization problem is therefore to minimize J, for each $(s, y; u)$, over all controls $u(t) \in U$. The function u which achieves this minimum is called an optimal control. In fact, the optimization problem with performance index as defined in equation (2.3) is called a Bolza problem. There are two other equivalent optimization problems, which are called Lagrange and Mayer problems [1, 11]. To clarify these definitions, the following example is described:

Example 2.1. *[12] The object is to find the optimal control which minimizes*

$$J = x^2(2),$$

when

$$\begin{cases} \dot{x}(t) = u(t), \\ x(0) = 1. \end{cases}$$

and $U = L^\infty([0, 2]; [-1, 1])$, $0 \leq t \leq 2$. *Since we have no running cost, our best course of action is to steer as fast as possible toward zero (where* $\psi(t, x)$ *is minimized). We are limited to velocities in the range* $[-1, 1]$ *but we have* 2 *time units to get to* 0. *So we can easily reach* 0. *In fact there are many optimal controls here such as:*

$$u(t) \equiv -\frac{1}{2},$$

and

$$u(t) = \begin{cases} -1, & 0 \leq t \leq 1, \\ 0, & t \geq 1, \end{cases}$$

If instead, $x(0) = -2$, *then the only optimal control would be* $u(t) \equiv 1$.

3 Stochastic Differential Equations

Stochastic differential equations are often written in the form of:

$$\begin{cases} dX(t) = f(t, X(t))dt + \sigma(t, X(t))dW(t), \\ X(s) = y, \end{cases}$$

which looks almost like an ordinary differential equation. However, as usual, the Ito differentials are not sensible mathematical objects in themselves; rather, we should see this expression as suggestive notation for the Ito process:

$$X(t) = y + \int_s^t f(\tau, X(\tau))d\tau + \int_s^t \sigma(\tau, X(\tau))dW(\tau),$$

If there exists a stochastic process X(t) that satisfies this equation, we say that it solves the stochastic differential equation.

Example 3.1. ***(Scalar linear equation)*** *Consider the scalar linear stochastic differential equation*

$$\begin{cases} dX(t) = aX(t)dt + bX(t)dW(t), \\ X(0) = y, \end{cases}$$

driven by a scalar Wiener process $W(t)$, *with* a *and* b *constants. This stochastic differential equation is said to have multiplicative noise. In fact, we can analytically solve this equation, the solution is [13]*

$$X(t) = y \exp\left(\left(a - \frac{1}{2}b^2\right)t + bW(t)\right),$$

4 Statement of Stochastic Optimal Control

Consider the stochastic system to be controlled

$$dX(t) = f(t, X(t), u(t))dt + b(t, X(t), u(t))dW(t), \tag{4.4}$$

with initial condition

$$X(s) = y, \tag{4.5}$$

where y is a given vector in $\mathbb{R}^n$. Also, $X(t) \in X \subset \mathbb{R}^n$ is the state process, $u(t) \in U \subset \mathbb{R}^m$ is the control process, $W(t)$ is a Wiener process, $f(t, X(t), u(t))$ is a drift, and $b(t, X(t), u(t))$ is diffusion [1, 6]. The optimal control rule μ, that determines the control u, is Markovian and is presented by

$$u(t) = \mu(t, X(t)),$$

and chosen so as to minimize

$$\min_u J(s, y; u)$$

where

$$J(s, y; u) = \mathbb{E}_{sy}\left[\int_s^T L(\tau, X(\tau), u(\tau))d\tau + \psi(X(T))\right]. \tag{4.6}$$

Now, Value function is defined as:

$$V(s, y) = \inf_{u \in U} \mathbb{E}_{sy}\left[\int_s^T L(\tau, X(\tau), u(\tau))d\tau + \psi(X(T))\right] = J(s, y; u^*),$$

i.e. V is the minimum cost achievable starting from initial condition $X(s) = y$, and $u^*(.)$ is the optimal control which achieves this minimum cost.

5 Dynamic Programming: Hamilton-Jacobi-Bellman Equation

The use of the principle of optimality, usually known as Dynamic Programming, to derive an equation for solving optimal control problem, was first proposed by Bellman [2]. In Dynamic Programming, a family of fixed initial point controls problem is considered. The minimum value of the performance index is considered as a function of this initial point. This function is called the value function. Whenever the value function is differentiable, it satisfies a nonlinear first order hyperbolic partial differential equation called the partial differential equation of dynamic programming. This equation is used for constructing a nonlinear optimal feedback control law [1, 6].

Lemma 5.1. *(Dynamic Programming Principle)*

$$V(s, y) = \inf_{u \in U} \mathbb{E}_{sy}\left[\int_s^{s+h} L(\tau, X(\tau), u(\tau))d\tau + V(s+h, X(s+h))\right]$$

where $X(t+h)$ is determined by u from stochastic differential equation (4.4) *with initial condition* (4.5)*.*

Proof. See [6].

Now, we introduce the concept of a backwards evolution operator:

Definition 5.1. *Given a stochastic process $X(t)$, let*

$$\mathcal{A}\phi = (\mathcal{A}\phi)(s,y) = \lim_{h\to 0}\frac{1}{h}\Big(\mathbb{E}_{sy}(\phi(s+h, X(s+h))) - \phi(s,y)\Big)$$

We call $\mathcal{A}$ the backwards evolution operator for the stochastic process $X(t)$. operator $\mathcal{A}$ acts on a domain $D(\mathcal{A})$ consisting of functions $\phi(s,y)$ for which the above limit exists for all s and y.

Lemma 5.2. *The backwards evolution operator associated with $X(t)$ generated by the Stochastic differential equation with fixed control $u(s) \equiv v$*

$$\begin{cases} dX(t) = f(t, X(t), v)dt + \sigma(t, X(t), v)dW(t) \\ X(s) \equiv y \end{cases}$$

is

$$\mathcal{A}^v\phi = \phi_t + \sum_{i=1}^{n} f_i(t, X, v)\phi_{X_i} + \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(t, X, v)\phi_{X_iX_j}$$

where $a = \sigma\sigma^{'}$.

Proof. See [6].

Definition 5.2. *(Dynkin's formula) For $s < t$*

$$\mathbb{E}_{sy}\,\phi(t, X(t)) - \phi(s,y) = E_{sy}\int_s^t \Big(\phi_t(\tau, X(\tau)) + \mathcal{A}(\tau)\psi(\tau, X(\tau))\Big)d\tau.$$

Now, for obtaining HJB equation we take the expected value of a criterion of Bolza type (4.6) where the control applied at time t using feedback control u is $u(t, X(t))$. In dynamic programming the optimal expected system performance is considered as a function of the initial data:

$$V(s,y) = \inf_u J(s, y; u).$$

An optimal feedback control law u^* has the property that $V(s,y) = J(s, y; u^*)$ for all (s,y). If u is given then by Dynkin formula with $\psi = V$ and $\mathcal{A} = \mathcal{A}^u$, we have the following equation for V:

$$V(s,y) = -\mathbb{E}_{sy}\left[\int_s^t \big(V_s + A^u(\tau)V\big)d\tau + V(t, X(t))\right], \tag{5.7}$$

Now, suppose that the controller uses u for times $s \le \tau \le t$ and uses an optimal control u^* after time s. Its expected performance can be no less than $V(t,x)$. Thus let

$$u_1(\tau, x) = \begin{cases} u(\tau, x) & \tau \le t \\ u^*(\tau, x) & \tau > t \end{cases}$$

by properties of conditional expectation,

$$J(s, y, u_1) = \mathbb{E}_{sy}\left[\int_s^t L(\tau, X(\tau), u(\tau))d\tau + J(t, X(t), u^*)\right].$$

Then

$$\begin{aligned} &V(s,y) \le J(s,y,u_1), \quad V(t,X(t)) = J(t,X(t),u^*), \\ &V(s,y) \le \mathbb{E}_{sy}\left[\int_s^t [L(\tau,X(\tau),u(\tau)]d\tau + V(t,X(t))\right]. \end{aligned} \tag{5.8}$$

Equality holds, if an optimal control law $u = u^*$ is used during $[s,t]$. Let us subtract (5.7) from (5.8) and divide by $t-s$. Since $X(s) = y$, we get $t \to s^+$, with $v = u(s,y)$:

$$0 \le V_s + A^v(s)V + L(s,y,v).$$

Equality holds if $v = u^*(s,y)$, where u^* is an optimal feedback control law. Thus, for V we have derived the continuous time dynamic programming equation of optimal stochastic control theory:

$$\begin{aligned} 0 &= \tfrac{\partial V}{\partial s} + \min_{u\in U}\left[L(s,X(s),v) + A^{u(t)}V(s,y)\right] \\ &= \tfrac{\partial V}{\partial s} + \min_{v(t)\in U}\left[L(s,y,v) + f(s,y,v).D_X V + \tfrac{1}{2}Trace\left(a(s,y,v)D_X^2 V\right)\right], \end{aligned}$$

where $D_X^2 V$ is Hessian of V and $Trace\left(a(s,y,v)D_X^2 V\right) = \sum_{i,j=1}^n a_{ij}(s,y,v)V_{X_iX_j}$. Now we introduce the function

$$H(s,y,D_XV,D_X^2V) = \min_{v(t)\in U}\left[L(s,y,v) + f(s,y,v).D_XV + \frac{1}{2}Trace\left(a(s,y,v)D_X^2V\right)\right],$$

with the boundary condition

$$V(T,X) = \psi(X),$$

In fact, we have shown that Vsolves the HJB equation:

$$(HJB) \quad \begin{cases} \frac{\partial V}{\partial t} + H(t,X,D_XV,D_X^2V) = 0 \\ V(T,X) = \psi(X) \end{cases}$$

Now it is clear how we might use this to solve a stochastic control problem; namely, given a control problem, we first solve HJB equation to obtain $V(s,y)$ and hopefully in the process divine what the optimal control $u^*(.)$ is. Next example clarifies the above concept:

Example 5.1. *[4] Suppose this stochastic differential equation is given*

$$\begin{cases} dX(t) = u(t)dt + dW(t), \\ X(0) = \sqrt{2}. \end{cases}$$

and our cost functional is

$$J(s,y;u) = E_{sy}\left[\frac{1}{2}\int_0^1 u^2(\tau)d\tau + \frac{1}{2}X^2(1))\right]$$

and assume that $U = L^\infty([0,t];\mathbb{R})$. That is, we basically can control the path entirely as we have direct control over the velocity, but we are hampered by a Brownian white noise. Ideally it makes sense to try to steer toward the lowest terminal cost, but we must weigh that against a running cost which is higher for higher velocities. Suppressing some of the math we have that

$$\mathcal{A}^v = v\frac{\partial}{\partial X} + \frac{1}{2}\frac{\partial^2}{\partial X^2}$$

and

$$H(t, X, D_x V, D_x^2 V) = \min_{v(t)\in U} \left\{ v\frac{\partial V}{\partial X} + \frac{1}{2}\frac{\partial^2 V}{\partial X^2} + \frac{1}{2}v^2 \right\} = \frac{1}{2}\frac{\partial^2 V}{\partial X^2} - \frac{1}{2}\left(\frac{\partial V}{\partial X}\right)^2 .$$

where the choice of v *which achieves the minimum is* $v = -\frac{\partial V}{\partial X}$ *(and so once we find our value function* $V(t,x)$ *we will therefore want our Markov control policy to be* $u^*(t,X) = -\frac{\partial V}{\partial X}(t,X)$*). Our Hamilton-Jacobi-Bellman equation is thus*

$$(HJB) \quad \begin{cases} \frac{\partial V}{\partial t} - \frac{1}{2}\left(\frac{\partial V}{\partial X}\right)^2 + \frac{1}{2}\frac{\partial^2 V}{\partial X^2} = 0 \\ V(1, X(1)) = \frac{1}{2}X^2(1) \end{cases}$$

The HJB equation has not analytical solution in general. We solve this equation via separation of variables by guessing a solution of the form $V(t,X) = K_1(t)X^2 + K_2(t)$*. Solving the resulting ordinary differential equation for* $V(t,X)$*, we find that*

$$V(t,X) = \frac{1}{4-2t}X^2 + \frac{1}{2}\ln(2-t),$$

and the exact solution for the performance index is $\frac{1}{2}(1+\ln(2))$*. Taking the partial derivatives of* $V(t,X)$ *and substituting the formula for our Markov control policy above we find that*

$$u^*(t,X) = -\frac{\partial V}{\partial X} = \frac{1}{t-2}X(t).$$

6 Stochastic Optimal Control Problems with Linear Control: A Financial Application

In this section we study problems which are linear in the control and illustrate a method to solve such problems. As mentioned before, only a small class of stochastic optimal control problems admits analytic solutions for the value function and the corresponding optimal strategies. Finite state Markov chain approximation and finite differences do not work well in the case where the problem is linear in the control. To investigate the property of presented method, we have used the same example as in [10]. Chavanasporn and Ewald [10] consider the situation where a firm is considering investing into a project, where the time upon completion is uncertain. The level of completion represents the state variable. Once the project is fully completed, an asset will be obtained with certainty and will lead to a financial benefit for the firm. In fact the situation they consider is that a firm is facing a decision problem of investing into a project. The project takes an uncertain time to complete. In this example, the investment rate takes value either a fixed rate of investment; say k, or zero and the HJB equation will consist of two differential equations. One differential equation for the case where the firm does not invest and another differential equation for the case where the firm invest [10]. The situation they consider is that a firm is facing a decision problem of investing into a project. The project takes an uncertain time to complete. Let $x \in [0,\,1]$ be the level of completion, where $x = 0$ means nothing has been done while $x = 1$ means the project is fully completed. Upon completion, the firm receives an asset whose value M is known with certainty. The resulting optimization problem is given as [10]:

$$V(x) = \max_{u\in[0,k]} \left\{ \mathbb{E}\left[e^{-r\tau}M - \int_0^\tau e^{-rt}u(t)x(t)dt \right] \right\}, \tag{6.9}$$

such that

$$dx(t) = \ (\mu + u(t)\ )x(t)dt + \sigma x(t)dW(t),\ x(0) = x_0,\ x \in [0,1]. \tag{6.10}$$

The firm maximizes (6.9) subject to (6.10). It has a fixed contribution rate μ and a choice of investment $u(t)$ which the firm can decide upon how much it should invest into the project with the maximum investment rate $k > 0$. The fixed contribution can be thought of as a rental or storage cost that the firm needs to pay, no matter what it invests into the project. The parameter $\sigma > 0$ denotes the volatility and $W(t)$ is a Wiener process. In the objective function, equation (6.10), M represents the obtained benefit of the firm from the asset, which results from the fully completed project whereas, $u(t)x(t)$ represents the investment cost [10]. The corresponding Hamilton-Jacobi-Bellman equation is given by

$$rV(x) = \max_{u\in[0,k]} \left\{ (\mu + u)xV'(x) + \frac{1}{2}\sigma^2 x^2 V''(x) - ux \right\}. \tag{6.11}$$

Due to equation (6.11), being linear in u, the rate of investment that maximizes the right hand side of (6.11) is always either zero or the maximum rate k:

$$u^* = \begin{cases} k & V'(x) - 1 \geq 0 \\ 0 & o.w. \end{cases}$$

Let $V_0(x)$ and $V_k(x)$ denote the value functions when the firm invest 0 and k respectively. The Hamilton-Jacobi-Bellman equations for $V_0(x)$ and $V_k(x)$ becomes

$$\begin{cases} rV_0(x) = \mu x V_0'(x) + \frac{1}{2}\sigma^2 x^2 V_0''(x), & V_0(0) = 0 & x < x^* \\ rV_k(x) = (\mu + k)xV_k'(x) + \frac{1}{2}\sigma^2 x^2 V_k''(x) - kx, & V_k(1) = M & x \geq x^* \end{cases} \tag{6.12}$$

Now, we solve this system of equation via separation of variables by guessing a solution of the form

$$\begin{cases} V_0(x) = A_1 x^{\beta_1}, & x < x^* \\ V_k(x) = A_2 x^{\beta_2} + A_3 x^{\beta_3} + A_4 x & x \geq x^* \end{cases} \tag{6.13}$$

Substituting (6.13) into (6.12) gives

$$\begin{cases} \beta_1 = \frac{-(\mu - \frac{1}{2}\sigma^2) + \sqrt{(\mu - \frac{1}{2}\sigma^2)^2 + 2r\sigma^2}}{\sigma^2}, \\ \beta_2 = \frac{-(\mu + k - \frac{1}{2}\sigma^2) + \sqrt{(\mu + k - \frac{1}{2}\sigma^2)^2 + 2r\sigma^2}}{\sigma^2}, \\ \beta_3 = \frac{-(\mu + k - \frac{1}{2}\sigma^2) - \sqrt{(\mu + k - \frac{1}{2}\sigma^2)^2 + 2r\sigma^2}}{\sigma^2}, \\ A_4 = \frac{k}{\mu + k - r}. \end{cases}$$

Also, A_1, A_2, A_3 and $x^*$are still to be determined. These parameters can be obtained by solving following system:

$$\begin{cases} V_0(x^*) = V_k(x^*), \\ V_0'(x^*) = V_k'(x^*), \\ V_0''(x^*) = V_k''(x^*), \\ V_k(1) = M. \end{cases} \tag{6.14}$$

For the purpose of illustration, the following parameters have been chosen [10]: $r = 0.04$, $\mu = 0.02$, $\sigma = 0.2$, $k = 0.1$and $M = 1$. In this case, we have:

$$\begin{cases} \beta_1 = \sqrt{2}, \\ \beta_2 = \frac{-5+\sqrt{33}}{2}, \\ \beta_3 = \frac{-5-\sqrt{33}}{2}, \\ A_4 = \frac{5}{4}, \end{cases}$$

Also, solution of system (6.14) yields:

$$\begin{cases} V_0(x) = 1.191453111x^{\sqrt{2}}, & x < 0.2837648727, \\ V_k(x) = -0.2500027233x^{\frac{-5+\sqrt{33}}{2}} + 0.0000027233x^{\frac{-5-\sqrt{33}}{2}} + \frac{5}{4}x, & x \geq 0.2837648727. \end{cases}$$

Figure 1 shows the value functions $V_0(x)$ and $V_k(x)$ as a function of x. According to the simulation, it is optimal for the firm to invest with the investment rate k when x is greater than or equal to the threshold $x^* = 0.2837648727$ and to momentarily stop investing if x is less than the threshold x^*.

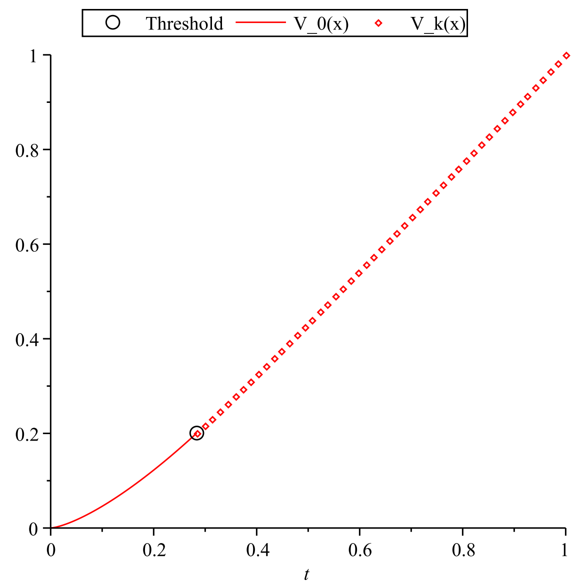

Fig. 1. the value function $V(x)$

7 Conclusion

Only a small class of stochastic optimal control problems admits analytic solutions for the value function and the corresponding optimal strategies. Two classical approaches in numerical solving stochastic optimal control problems are to use finite differences to solve the corresponding Hamilton- Jacobi-Bellman equation or finite state Markov chain approximation, to reduce the problem to a finite dimensional problem, which can be solved by using vector-matrix operations. Both approaches however have their limitations and do not work well in the case where the problem is linear in the control. In this case optimal strategy switches between two modes, a maximum and a minimum control mode, and the Hamilton-Jacobi-Bellman equation effectively breaks down into two differential equations, which are linked at the threshold, where it is optimal to switch. In this paper, we investigate stochastic optimal control problems and Bellman's dynamic programming method (Hamilton-Jacobi-Bellman equation). Dynamic programming method represents the most known method for solving optimal control problems. Also, optimal control problems with linear control are study in this work. Finally, we simulate a financial example to highlight the applications of stochastic optimal control problems.

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